Math 245C Lecture 15 Notes

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1 Approximation of L^p Functions by Convolutions with Scaled Mollifiers

Today's lecture was given by a guest lecturer.

1.1 Approximation of L^p functions by convolutions with scaled mollifiers

Theorem 1.1. *Suppose $|\phi(x)| \leq C(1+|x|)^{-n-\varepsilon}$ for some $C, \varepsilon > 0$ (so $\phi \in L^1(\mathbb{R}^d)$), and let $\int_{\mathbb{R}^d} \phi(x)\,dx = a$. If $f \in L^p$ with $1 \leq p \leq \infty$, then $f * \phi_t(x) \to af(x)$ as $t \to 0^+$ for every x in the Lebesgue set of f.*

Remark 1.1. This implies that $f * \phi_t(x) \to af(x)$ for a.e. x and for every x for which f is continuous

Proof. If x is in the Lebesgue set of f, for any $\delta > 0$, there exists an $\eta > 0$ such that

$$\int_{B_r} |f(x-y) - f(x)|\,dy \leq \delta r^n, \qquad \forall r \leq \eta.$$

In other words, $\lim_{r \to 0^+} \frac{1}{|B_r|} \int_{B_r} |f(x-y) - f(x)|\,dy = 0$. We have

$$\begin{aligned} |f * \phi_t(x) - af(x)| &= \left| \int_{\mathbb{R}^d} f(x-y)\phi_t(y) - f(x)\phi_t(y)\,dy \right| \\ &= \int_{\mathbb{R}^d} |\phi_t(y)||f(x-y) - f(x)|\,dy \\ &= \underbrace{\int_{B_y} |\phi_t(y)||f(x-y) - f(x)|\,dy}_{I_1} + \underbrace{\int_{B_y} |\phi_t(y)||f(x-y) - f(x)|\,dy}_{I_2}. \end{aligned}$$

We claim that $I_a \leq A\delta$ for some A independent of t and that $I_2 \to 0$ as $t \to 0^+$. If the claim holds, then

$$\lim_{t \to 0^+} |f * \phi_t(x) - af(x)| \leq \lim_{t \to 0^+} I_1 \leq A\delta$$

Letting $\delta \to 0$,

$$\lim_{f \to 0^+} f * \phi_t(x) = af(x).$$

To estimate I_1, let $K \in \mathbb{Z}$ be such that $2^K \leq \eta/t \leq 2^{K+1}$ if $\eta/t \geq 1$ and $K = 0$ if $\eta/t < 1$. We view the ball B_y as the union of $B_{2^{1-k}\eta} \setminus B_{2^{-k}\eta}$ for $k = 1, 2, 3, \ldots, K$ and the ball $B_{2^{-K}\eta}$. We have a few cases:

1. On $B_{2^{1-k}\eta} \setminus B_{2^{-k}\eta}$ for $k = 1, \ldots, K$,

$$|\phi_t(y)| = t^{-n}\phi(t^{-1}y)| \leq Ct^{-n}(1 + |t^{-1}y|)^{-n-\varepsilon} \leq Ct^{-n}(1 + |t^{-1}2^{-k}\eta|)^{-n-\varepsilon}.$$

2. On $B_{2^{-K}\eta}$,

$$|\phi_t(y)| = t^{-n}|\phi(t^{-1}y)| \leq Ct^{-n}.$$

So

$$\begin{aligned}
I_1 &= \int_{B_\eta} |\phi_t(y)||f(x-y) - f(x)|\,dy \\
&= \sum_{k=1}^{K} \int_{B_{2^{1-k}\eta} \setminus B_{2^{-k}\eta}} |\phi_t(y)||f(x-y) - f(x)|\,dy + \int_{B_{2^{-K}\eta}} |\phi_t(y)||f(x-y) - f(x)|\,dy \\
&\leq \sum_{k=1}^{K} \left(\int_{B_{2^{1-k}\eta}} |f(x-y) - f(x)|\,dy \right) Ct^{-n}|e^{-1}2^{-k}\eta|^{-n-\varepsilon} \\
&\qquad + \left(\int_{B_2^{-K}\eta} |f(x-y) - f(x)|\,dy \right) Ct^{-n} \\
&\leq \left(\sum_{k=1}^{K} Ct^{-n}|t^{-1}2^{-k}\eta|^{-n-\varepsilon}\delta(2^{1-k}\eta)^n \right) + Ct^{-n}\delta(2^{-K}\eta)^n \\
&= C\delta \left(\frac{t}{\eta}\right)^{\varepsilon} 2^n \sum_{k=1}^{K} 2^{k\varepsilon} + C_\delta \left(\frac{2^{-K}\eta}{t}\right)^n \\
&= C\delta 2^n \left(\frac{t}{\eta}\right)^{\varepsilon} \frac{2^{(K+1)\varepsilon} - 2^{\varepsilon}}{2^{\varepsilon} - 1} + C_\delta \left(\frac{2^{-K}\eta}{t}\right)^n
\end{aligned}$$

Use the inequality defining K:

$$\begin{aligned}
&\leq C\delta 2^n 2^{-K\varepsilon} \frac{2^{(K+1)\varepsilon} - 2^{\varepsilon}}{2^{\varepsilon} - 1} + C\delta 2^n \\
&= \underbrace{2^n C(2^{\varepsilon}(2^{\varepsilon} - 1) + 1)}_{:=A}\,\delta.
\end{aligned}$$

To estimate I_2, we have, using Hölder's inequality,

$$I_n \leq \int_{B_\eta^c} (|f(x-y)| + |f(x)|)\, |\phi_t|(y) \leq \|f\|_{p'} \| \|\mathbb{1}_{B_\eta^c}\phi_t\|_p + |f(x)| \|\mathbb{1}_{B_\eta^c}\phi_t\|.$$

We split into cases:

1. $p' = \infty$: Then

$$\|\mathbb{1}_{B_\eta^c}\phi_t\|_{p'} \leq Ct^{-n}(1 + t^{-1}\eta)^{-n-\varepsilon} = Ct^\varepsilon (t+\eta)^{-n-\varepsilon} \leq Ct^\varepsilon \eta^{-n-\varepsilon}.$$

2. $1 \leq p' < \infty$:

$$\begin{aligned}
&\|\mathbb{1}_{B_\eta^c}\phi_t\|_{p'} \\
&\qquad = \int_{B_\eta} t^{-np'} |\phi(t^{-1}y))|^{p'}\, dy \\
&\qquad = t^{n(1-p')} \int_{B_{\eta/t}^c} |\phi(z)|^p\, dz \\
&\qquad \leq Ct^{n(1-p')} \int_{B_{\eta/t}^c} \left[(1+|z|)^{-n-\varepsilon}\right]^{p'}\, dz \\
&\qquad = Ct^{n(1-p')} \left(\frac{\eta}{t}\right)^{n-(n-\varepsilon)p'} \\
&\qquad \leq Ct^{\varepsilon p'},
\end{aligned}$$

which goes to 0 as $t \to 0^+$. □

Suppose we want to show that C_c^∞ is dense in L^p. Then we let $f_n = f\mathbb{1}_{B_n}$, so $f_n \to f$ in L^p. The idea is then that $f_n * \phi_t \to f - n$ as $t \to 0^+$, so $f_n * \phi_t \in C_c^\infty$ approximates f in L^p.